POWER REDUCTIVITY OVER AN ARBITRARY BASE

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Abstract. We prove a straight generalization to an arbitrary base of Mumford's conjecture on Chevalley groups over fields. We thus obtain the first fundamental theorem of invariant theory (often referred to as Hilbert's fourteenth problem) over an arbitrary Noetherian ring. We also prove results on the Grosshans graded of an algebra in the same generality.

1. Introduction

The following statement may seem quite well known:

Theorem 1. *Let* $\mathbf{k}$ *be a Dedekind ring and let* G *be a Chevalley group scheme over* $\mathbf{k}$. *Let* A *be a finitely generated* $\mathbf{k}$*-algebra on which* G *acts rationally through algebra automorphisms. The subring of invariants* A^G *is then a finitely generated* $\mathbf{k}$*-algebra.*

Indeed, R. Thomason proved [20, Theorem 3.8] the statement for any Noetherian Nagata ring $\mathbf{k}$. Thomason's paper deals with quite a different theme, that is the existence of equivariant resolutions by free modules. Thomason thus solves a conjecture of Seshadri [18, question 2 p.268.] The affirmative answer to the question is explained to yield Theorem 1 in the same paper [18, Theorem 2 p.263]. However, the definition of geometric reductivity does not suit well an arbitrary base. Seshadri does not follow the formulation in Mumford's book's introduction [14, Preface], and uses polynomials instead [18, Theorem 1 p.244]. This use of a dual in the formulation seems to be what requires Thomason's result [20, Corollary 3.7]. One can rather go back to the original formulation in terms of symmetric powers as illustrated by the following:

Definition 2. Let $\mathbf{k}$ be a ring and let G be an algebraic group over $\mathbf{k}$. The group G is *power-reductive* over $\mathbf{k}$ if the following holds.

Property (Power reductivity)**.** *Let* L *be a cyclic* $\mathbf{k}$*-module with trivial* G*-action. Let* M *be a rational* G*-module, and let* φ *be a* G*-module map from* M *onto* L. *Then there is a positive integer* d *such that the* d*-th symmetric power of* φ *induces a surjection:*

$$(S^d M)^G \to S^d L.$$

We show in Section 2 that power-reductivity holds for Chevalley group schemes G, without assumption on the commutative ring $\mathbf{k}$. Note that this version of reductivity is exactly what is needed in Nagata's treatment of finite generation of invariants. We thus obtain:

Theorem 3. *Let* $\mathbf{k}$ *be a Noetherian ring and let* G *be a Chevalley group scheme over* $\mathbf{k}$. *Let* A *be a finitely generated* $\mathbf{k}$*-algebra on which* G *acts rationally through algebra automorphisms. The subring of invariants* A^G *is then a finitely generated* $\mathbf{k}$*-algebra.*

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The author acknowledges the hospitality and support of CRM Barcelona during the tuning of the paper.
** The project started in Nantes, the author being the first visitor in the MatPyL program of CNRS' FR 2962 "Mathématiques des Pays de Loire".

Section 4 generalizes to an arbitrary (Noetherian) base Grosshans' results on his filtration. These are basic tools for obtaining a cohomological finite generation statement, such as the conjecture in [12]. Section 3 presents results of use in the other sections 4 and 2.

2. Power reductivity, Mumford's conjecture revisited and Hilbert's 14th over an arbitrary base

2.1. **Consequences.** We start by listing consequences of power reductivity, as defined in the introduction (Definition 2). To deal with the strong form of integrality we encounter, we find it convenient to make the following definition.

Definition 4. A morphism of $\mathbf{k}$-algebras: $\phi : S \to R$ is *power-surjective* if every element of R has a power in the image of ϕ. It is *universally power-surjective* if for every $\mathbf{k}$-algebra A, the morphism of $\mathbf{k}$-algebras $A \otimes \phi$ is power-surjective, that is: for every $\mathbf{k}$-algebra A, for every x in $A \otimes R$, there is a positive integer n so that x^n lies in the image of $A \otimes \phi$.

If $\mathbf{k}$ contains a field, one does not need arbitrary positive exponents n, but only powers of the characteristic exponent of $\mathbf{k}$ (compare [19, Lemma 2.1.4, Exercise 2.1.5]). Thus if $\mathbf{k}$ is a field of characteristic zero, any universally power-surjective morphism of $\mathbf{k}$-algebras is surjective.

Proposition 5. *Let $\mathbf{k}$ be a ring and let G be a power-reductive algebraic group over $\mathbf{k}$. Let A be a finitely generated $\mathbf{k}$-algebra on which G acts rationally through algebra automorphisms. If J is an invariant ideal in A, the map induced by reducing mod J:*

$$A^G \to (A/J)^G$$

is power-surjective.

As explained in in [13, Theorem 2.8], when $\mathbf{k}$ is a field, the property (Int) that $(A/J)^G$ is integral over the image of A^G for every A and J, is equivalent to geometric reductivity, which is equivalent to power-reductivity by [19, Lemma 2.4.7 p. 23]. For an example over $\mathbb{Z}$, see 2.8.2.

Theorem 6 (Hilbert's fourteenth)**.** *Let $\mathbf{k}$ be a Noetherian ring and let G be an algebraic group over $\mathbf{k}$. Let A be a finitely generated $\mathbf{k}$-algebra on which G acts rationally through algebra automorphisms. If G is power-reductive, then the subring of invariants A^G is a finitely generated $\mathbf{k}$-algebra.*

Proof. We apply [19, p. 23–25]. It relies entirely on the statement [19, Lemma 2.4.7 p. 23] that, for a surjective G-map $\phi : A \to B$ of $\mathbf{k}$-algebras, the induced map on invariants $A^G \to B^G$ is power-surjective. To prove that power reductivity implies this lemma, consider an invariant b in B, take for L the cyclic module $\mathbf{k}.b$ et for M any submodule of A such that $\phi(M) = L$. We conclude with a commuting diagram:

$$\begin{array}{ccccc}
(S^dM)^G & \longrightarrow & (S^dA)^G & \longrightarrow & A^G \\
\downarrow & & \downarrow & & \downarrow \\
S^dL & \longrightarrow & (S^dB)^G & \longrightarrow & B^G.
\end{array}$$

□

Theorem 7 (Hilbert's fourteenth for modules)**.** *Let $\mathbf{k}$ be a Noetherian ring and let G be a power-reductive algebraic group over $\mathbf{k}$. Let A be a finitely generated $\mathbf{k}$-algebra on which G acts rationally, and let M be a Noetherian A-module, with an equivariant structure map $A \otimes M \to M$. If G is power-reductive, then the module of invariants M^G is Noetherian over A^G.*

Proof. As in [13, 2.2], consider the symmetric algebra of M on A. □

2.2. The rest of the section deals with the following generalization of the Mumford conjecture.

Theorem 8 (Mumford conjecture). *A Chevalley group scheme is power-reductive for every base.*

By a Chevalley group scheme over $\mathbb{Z}$, we mean a connected split reductive algebraic $\mathbb{Z}$-group $G_{\mathbb{Z}}$, and, by a Chevalley group scheme over a ring $\mathbf{k}$, we mean an algebraic $\mathbf{k}$-group $G = G_{\mathbf{k}}$ obtained by base change from such a $G_{\mathbb{Z}}$.

We want to establish the following:

Property. *Let $\mathbf{k}$ be a commutative ring. Let L be a cyclic $\mathbf{k}$-module with trivial G-action. Let M be a rational G-module, and let φ be a G-module map from M onto L. Then there is a positive integer d such that the d-th symmetric power of φ induces a surjection:*

$$(S^dM)^G \to S^dL.$$

2.3. **Reduction to local rings.** We first reduce to the case of a local ring. For each positive integer d, consider the ideal in $\mathbf{k}$ formed by those scalars which are hit by an invariant in $(S^dM)^G$, and let:

$$\mathfrak{I}_d(\mathbf{k}) := \{x \in \mathbf{k} \mid \exists m \in \mathbb{N},\ x^m.S^dL \subset S^d\varphi((S^dM)^G)\}$$

be its radical. Note that these ideals form a monotone family: if d divides d', then $\mathfrak{I}_d$ is contained in $\mathfrak{I}_{d'}$. We want to show that $\mathfrak{I}_d(\mathbf{k})$ equals $\mathbf{k}$ for some d. To that purpose, it is enough to prove that for each maximal ideal $\mathfrak{M}$ in $\mathbf{k}$, the localized $\mathfrak{I}_d(\mathbf{k})_{(\mathfrak{M})}$ equals the local ring $\mathbf{k}_{(\mathfrak{M})}$ for some d. Notice that taking invariants commutes with localization. Indeed the whole Hochschild complex does and localization is exact. As a result, the localized $\mathfrak{I}_d(\mathbf{k})_{(\mathfrak{M})}$ is equal to the ideal $\mathfrak{I}_d(\mathbf{k}_{(\mathfrak{M})})$. This shows that it is enough to prove the property for a local ring $\mathbf{k}$.

For the rest of this proof, $\mathbf{k}$ denotes a local ring with residual characteristic p.

2.4. **Reduction to cohomology.** As explained in Section 2.7, we may assume that G is semisimple simply connected. Replacing M if necessary by a submodule that still maps onto L, we may assume that M is finitely generated. We then reduce the desired property to cohomological algebra. To that effect, if X is a G-module, consider the evaluation map on the identity $\mathrm{id}_X\colon \mathrm{Hom}_{\mathbf{k}}(X,X)^\# \to \mathbf{k}$ (we use $V^\#$ to indicate the dual module $\mathrm{Hom}_{\mathbf{k}}(V,\mathbf{k})$ of a module V). If X is $\mathbf{k}$-free of finite rank d, then $S^d(\mathrm{Hom}_{\mathbf{k}}(X,X)^\#)$ contains the determinant. The determinant is G-invariant, and its evaluation at id_X is equal to 1. Let b a $\mathbf{k}$-generator of L and consider the composite:

$$\psi : \mathrm{Hom}_{\mathbf{k}}(X,X)^\# \to \mathbf{k} \to \mathbf{k}.b = L.$$

Its d-th power $S^d\psi$ sends the determinant to b^d. Suppose further that ψ lifts to M by a G-equivariant map. Then, choosing d to be the $\mathbf{k}$-rank of X, the d-th power of the resulting map $S^d(\mathrm{Hom}_{\mathbf{k}}(X,X)^\#) \to S^dM$ sends the determinant to a G-invariant in S^dM, which is sent to b^d through $S^d\varphi$. This would establish the property.

$$\begin{tikzcd} & \mathrm{Hom}_{\mathbf{k}}(X,X)^\# \arrow[dl, dashed] \arrow[d, "\psi"] \\ M \arrow[r, "\varphi"'] & L \end{tikzcd}$$

The existence of a lifting would follow from the cancellation of the extension group:

$$\mathrm{Ext}^1_G((\mathrm{Hom}_{\mathbf{k}}(X,X)^\#, \mathrm{Ker}\varphi),$$

or, better, from the cancellation of all positive degree Ext-groups (or acyclicity).

Inspired by the proof of the Mumford conjecture in [5, (3.6)], we choose X to be an adequate Steinberg module. To make this choice precise, we need notations, essentially borrowed from [5, 2].

2.5. **Notations.** We decide as in [10], and contrary to [11] and [5], that the roots of the standard Borel subgroup B are negative. The opposite Borel group B^+ of B will thus have positive roots. We also fix a Weyl group invariant inner product on the weight lattice $X(T)$. Thus we can speak of the length of a weight.

For a weight λ in the weight lattice, we denote by λ as well the corresponding one-dimensional rational B-module (or sometimes B^+-module), and by ∇_λ the costandard module (Schur module) $\mathrm{ind}_B^G\lambda$ induced from it. Dually, we denote by Δ_λ the standard module (Weyl module) of highest weight λ. So $\Delta_\lambda = \mathrm{ind}_{B^+}^G(-\lambda)^\#$. We shall use that, over $\mathbb{Z}$, these modules are $\mathbb{Z}$-free [10, II Ch. 8].

We let ρ be half the sum of the positive roots of G. It is also the sum of the fundamental weights. As G is simply connected, the fundamental weights are weights of B.

Let p be the characteristic of the residue field of the local ring $\mathbf{k}$. When p is positive, for each positive integer r, we let the weight σ_r be $(p^r - 1)\rho$. When p is 0, we let σ_r be $r\rho$. Let St_r be the G-module $\nabla_{\sigma_r} = \mathrm{ind}_B^G\sigma_r$; it is a usual Steinberg module when $\mathbf{k}$ is a field of positive characteristic.

2.6. We shall use the following combinatorial lemma:

Lemma 9. *Let R be a positive real number. If r is a large enough integer, for all weights μ of length less than R, $\sigma_r + \mu$ is dominant.*

So, if r is a large enough integer to satisfy the condition in Lemma 9, for all G-modules M with weights that have length less than R, all the weights in $\sigma_r \otimes M$ are dominant. Note that in the preceding discussion, the G-module M is finitely generated. Thus the weights of M, and hence of $\mathrm{Ker}\varphi$, are bounded. Thus, Theorem 8 is implied by the following proposition.

Proposition 10. *Let R be a positive real number, and let r be as in Lemma 9 . For all local rings $\mathbf{k}$ with characteristic p residue field, for all G-module N with weights of length less than R, and for all positive integers n:*

$$\mathrm{Ext}^n_G((\mathrm{Hom}_{\mathbf{k}}(St_r, St_r)^\#, N) = 0\ .$$

Proof. First, the result is true when $\mathbf{k}$ is a field. Indeed, we have chosen St_r to be a self-dual Steinberg module, so, for each positive integer n:

$$\mathrm{Ext}^n_G((\mathrm{Hom}_{\mathbf{k}}(St_r, St_r)^\#, N) = \mathrm{H}^n(G, St_r \otimes St_r \otimes N) = \mathrm{H}^n(B, St_r \otimes \sigma_r \otimes N).$$

Cancellation follows by [5, Corollary (3.3')] or the proof of [5, Corollary (3.7)].

Suppose now that N is defined over $\mathbb{Z}$ by a free $\mathbb{Z}$-module, in the following sense: $N = N_\mathbb{Z} \otimes_\mathbb{Z} V$ for a $\mathbb{Z}$-free $G_\mathbb{Z}$-module $N_\mathbb{Z}$ and a $\mathbf{k}$-module V with trivial G action. We then use the universal coefficient theorem [3, A.X.4.7] (see also [10, I.4.18]) to prove acyclicity in this case.

Specifically, let us note $Y_\mathbb{Z} := \mathrm{Hom}_\mathbb{Z}((St_r)_\mathbb{Z}, (St_r)_\mathbb{Z}) \otimes N_\mathbb{Z}$, so that, using base change (Proposition 11 for $\lambda = \sigma_r$):

$$\mathrm{Ext}^n_G((\mathrm{Hom}_{\mathbf{k}}(St_r, St_r)^\#, N) = \mathrm{H}^n(G, Y_\mathbb{Z} \otimes V).$$

This cohomology is computed [6, II.3] (see also [10, I.4.16]) by taking the homology of the Hochschild complex $C(G, Y_\mathbb{Z} \otimes V)$. This complex is isomorphic to the complex obtained by tensoring with V the integral Hochschild complex $C(G_\mathbb{Z}, Y_\mathbb{Z})$. Since the latter is a complex of torsion-free abelian groups, we deduce, by the universal coefficient theorem applied to tensoring with a characteristic p field k, and the cancellation for the case of such a field, that: $\mathrm{H}^n(G_\mathbb{Z}, Y_\mathbb{Z}) \otimes k = 0$, for all positive n. We apply this when k is the residue field of $\mathbb{Z}_{(p)}$; note that if $p = 0$ the residue field k is just $\mathbb{Q}$. Since the cohomology $\mathrm{H}^n(G_\mathbb{Z}, Y_\mathbb{Z})$ is finitely generated [10, B.6], the Nakayama lemma implies that: $\mathrm{H}^n(G_\mathbb{Z}, Y_\mathbb{Z}) \otimes \mathbb{Z}_{(p)} = 0$, for all positive n. And $\mathrm{H}^n(G_\mathbb{Z}, Y_\mathbb{Z}) \otimes \mathbb{Z}_{(p)} = \mathrm{H}^n(G_\mathbb{Z}, Y_\mathbb{Z} \otimes \mathbb{Z}_{(p)})$ because localization is exact. The complex $C(G_\mathbb{Z}, Y_\mathbb{Z} \otimes \mathbb{Z}_{(p)})$ is a complex of torsion-free $\mathbb{Z}_{(p)}$-modules, we thus can apply the universal coefficient theorem to tensoring with V. The cancellation of $\mathrm{H}^n(G, Y_{\mathbb{Z}} \otimes \mathbb{Z}_{(p)} \otimes V) = \mathrm{H}^n(G, Y_{\mathbb{Z}} \otimes V)$ follows.

For the general case, we proceed by descending induction on the highest weight of N. To perform the induction, we first choose a total order on weights of length less than R, that refines the usual dominance order of [10, II 1.5]. Initiate the induction with $N = 0$. For the induction step, consider the highest weight μ in N and let N_μ be its weight space. By the preceding case, we obtain vanishing for $\Delta_{\mu_{\mathbb{Z}}} \otimes_{\mathbb{Z}} N_\mu$. Now, by Proposition 16, $\Delta_{\mu_{\mathbb{Z}}} \otimes_{\mathbb{Z}} N_\mu$ maps to N, and the kernel and the cokernel of this map have lower highest weight. By induction, they give vanishing cohomology. Thus $\mathrm{Hom}_{\mathbf{k}}(St_r, St_r) \otimes N$ is in an exact sequence where three out of four terms are acyclic, hence it is acyclic. $\square$

This concludes the proof of Theorem 8.

2.7. **Reduction to simply connected group schemes.** Let $Z_{\mathbb{Z}}$ be the center of $G_{\mathbb{Z}}$ and let Z be the corresponding subgroup of G. It is a diagonalisable group scheme, so $M^Z \to L$ is also surjective. We may replace M with M^Z and G with G/Z, in view of the general formula $M^G = (M^Z)^{G/Z}$, see [10, I 6.8(3)]. So now G has become semisimple, but of adjoint type rather than simply connected type. So choose a simply connected Chevalley group scheme $\tilde{G}_{\mathbb{Z}}$ with center $\tilde{Z}_{\mathbb{Z}}$ so that $\tilde{G}_{\mathbb{Z}}/\tilde{Z}_{\mathbb{Z}} = G_{\mathbb{Z}}$. We may now replace G with $\tilde{G}$.

2.8. **Examples.**

2.8.1. Consider the group SL_2 acting on 2×2 matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ by conjugation. Let L be the line of homotheties in $M := M_2(\mathbb{Z})$. The restriction: $M^\# \to L^\#$ extends to

$$\mathbb{Z}[M] = \mathbb{Z}[a, b, c, d] \to \mathbb{Z}[\lambda] = \mathbb{Z}[L].$$

Taking SL_2-invariants:

$$\mathbb{Z}[a, b, c, d]^{\mathrm{SL}_2} = \mathbb{Z}[t, D] \to \mathbb{Z}[\lambda],$$

the trace $t = a + c$ is sent to 2λ, so λ does not lift to an invariant in $M^\#$. The determinant $D = ad - bc$ is sent to λ^2, illustrating power reductivity of SL_2.

2.8.2. Similarly, let SL_2 act on $\mathbb{Z}[sl_2] = \mathbb{Z}[X, H, Y]$ such that $u(a) := \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$ sends X, H, Y respectively to $X + aH - a^2Y, H - 2aY, Y$. The mod 2 invariant H does not lift to an integral invariant, but $H^2 + 4XY$ is an integral invariant, and it reduces to its square H^2.

2.8.3. Consider the group U of 2×2 upper triangular matrices with diagonal 1: this is just an additive group. Let it act on M with basis $\{x, y\}$ by linear substitutions: $u(a)$ sends x, y respectively to $x, ax + y$. Sending x to 0 defines $M \to L$, and since $(S^*M)^U = \mathbb{K}[x]$, power reductivity fails.

3. Generalities

This section collects known results over an arbitrary base, their proof, and correct proofs of known results over fields, for use in the other sections. The part up to subsection 3.3 is used, and referred to, in the previous section.

3.1. **Notations.** Let G be a semisimple Chevalley group scheme over the commutative ring $\mathbf{k}$. We keep the notations of Section 2.5. In particular, the standard parabolic B has negative roots. Its standard torus is T, its unipotent radical is U. The opposite Borel B^+ has positive roots and its unipotent radical is U^+. For a standard parabolic subgroup P its unipotent radical is $R_u(P)$. For a weight λ in $X(T)$, $\nabla_\lambda = \mathrm{ind}_B^G \lambda$ and $\Delta_\lambda = \mathrm{ind}_{B^+}^G(-\lambda)^\#$.

3.2. We first recall base change for costandard modules.

Proposition 11. *Let λ be a weight, and denote also by $\lambda = \lambda_{\mathbb{Z}} \otimes \mathbf{k}$ the B-module $\mathbf{k}$ with action by λ. For any ring $\mathbf{k}$, there is a natural isomorphism:*

$$\mathrm{ind}_{B_{\mathbb{Z}}}^{G_{\mathbb{Z}}} \lambda_{\mathbb{Z}} \otimes \mathbf{k} \cong \mathrm{ind}_B^G \lambda$$

In particular, $\mathrm{ind}_B^G \lambda$ is nonzero if and only if λ is dominant.

Proof. First consider the case when λ is not dominant. Then $\mathrm{ind}_B^G \lambda$ vanishes when $\mathbf{k}$ is a field [10, II.2.6], so both $\mathrm{ind}_{B_{\mathbb{Z}}}^{G_{\mathbb{Z}}} \lambda_{\mathbb{Z}}$ and the torsion in $R^1\mathrm{ind}_{B_{\mathbb{Z}}}^{G_{\mathbb{Z}}} \lambda_{\mathbb{Z}}$ must vanish. Then $\mathrm{ind}_B^G \lambda$ vanishes as well for a general $\mathbf{k}$ by the universal coefficient theorem.

In the case when λ is dominant, $R^1\mathrm{ind}_{B_{\mathbb{Z}}}^{G_{\mathbb{Z}}} \lambda_{\mathbb{Z}}$ vanishes by Kempf's theorem [10, II 8.8(2)]. Thus, by [10, I.4.18b)]: $\mathrm{ind}_{B_{\mathbb{Z}}}^{G_{\mathbb{Z}}} \lambda_{\mathbb{Z}} \otimes \mathbf{k} \cong \mathrm{ind}_B^G \lambda$. □

Proposition 12 (Tensor identity for weights)**.** *Let λ be a weight, and denote again by λ the B-module $\mathbf{k}$ with action by λ. Let N be a G-module. There is a natural isomorphism:*

$$\mathrm{ind}_B^G(\lambda \otimes N) \cong (\mathrm{ind}_B^G \lambda) \otimes N.$$

Remark 13. The case when N is $\mathbf{k}$-flat is covered by [10, I.4.8]. We warn the reader against Proposition I.3.6 in the 1987 first edition of the book. Indeed, suppose we always had $\mathrm{ind}_B^G(M \otimes N) \cong (\mathrm{ind}_B^G M) \otimes N$. Take $\mathbf{k} = \mathbb{Z}$ and $N = \mathbb{Z}/p\mathbb{Z}$. The universal coefficient theorem would then imply that $R^1\mathrm{ind}_B^G M$ never has torsion. Thus $R^i\mathrm{ind}_B^G M$ would never have torsion for positive i. It would make [1, Cor. 2.7] contradict the Borel–Weil–Bott theorem.

Proof. Recall that for a B-module M one may define $\mathrm{ind}_B^G(M)$ as $(\mathbf{k}[G] \otimes M)^B$, where $\mathbf{k}[G] \otimes M$ is viewed as a $G \times B$-module with G acting by left translation on $\mathbf{k}[G]$, B acting by right translation on $\mathbf{k}[G]$, and B acting the given way on M. Let N_{triv} denote N with trivial B action. There is a B-module isomorphism $\psi : \mathbf{k}[G] \otimes \lambda \otimes N \to \mathbf{k}[G] \otimes \lambda \otimes N_{\mathrm{triv}}$, given in non-functorial notation by:

$$\psi(f \otimes 1 \otimes n) : x \mapsto f(x) \otimes 1 \otimes xn.$$

So ψ is obtained by first applying the comultiplication $N \to \mathbf{k}[G] \otimes N$, then the multiplication $\mathbf{k}[G] \otimes \mathbf{k}[G] \to \mathbf{k}[G]$. It sends $(\mathbf{k}[G] \otimes \lambda \otimes N)^B$ to $(\mathbf{k}[G] \otimes \lambda \otimes N_{\mathrm{triv}})^B = (\mathbb{Z}[G_{\mathbb{Z}}] \otimes_{\mathbb{Z}} \lambda_{\mathbb{Z}} \otimes_{\mathbb{Z}} N_{\mathrm{triv}})^B$. Now recall from the proof of Proposition 11 that the torsion in $R^1\mathrm{ind}_{B_{\mathbb{Z}}}^{G_{\mathbb{Z}}} \lambda_{\mathbb{Z}}$ vanishes. By the universal coefficient theorem we get that $(\mathbb{Z}[G_{\mathbb{Z}}] \otimes_{\mathbb{Z}} \lambda_{\mathbb{Z}} \otimes_{\mathbb{Z}} N_{\mathrm{triv}})^B$ equals $(\mathbf{k}[G] \otimes \lambda)^B \otimes N_{\mathrm{triv}}$. To make these maps into G-module maps, one must use the given G-action on N as the action on N_{triv}. So B acts on N, but not N_{triv}, and for G it is the other way around. One sees that $(\mathbf{k}[G] \otimes \lambda)^B \otimes N_{\mathrm{triv}}$ is just $(\mathrm{ind}_B^G \lambda) \otimes N$. □

Proposition 14. *For a G-module M, there are only dominant weights in M^{U^+}.*

Proof. Let λ be a nondominant weight. Instead we show that $-\lambda$ is no weight of M^U, or that $\mathrm{Hom}_B(-\lambda, M)$ vanishes. By the tensor identity of Proposition 12:

$$\mathrm{Hom}_B(-\lambda, M) = \mathrm{Hom}_B(k, \lambda \otimes M) = \mathrm{Hom}_G(k, \mathrm{ind}_B^G(\lambda \otimes M)) = \mathrm{Hom}_G(k, \mathrm{ind}_B^G \lambda \otimes M)$$

which vanishes by Proposition 11. □

Proposition 15. *Let λ be a dominant weight. The restriction or evaluation map $\mathrm{ind}_B^G \lambda \to \lambda$ to the weight space of weight λ is a T-module isomorphism.*

Proof. Over fields of positive characteristic this is a result of Ramanathan [11, A.2.6]. It then follows over $\mathbb{Z}$ by the universal coefficient theorem applied to the complex $\mathrm{ind}_{B_{\mathbb{Z}}}^{G_{\mathbb{Z}}} \lambda_{\mathbb{Z}} \to \lambda_{\mathbb{Z}} \to 0$. For a general $\mathbf{k}$, apply proposition 11. □

Proposition 16 (Universal property of Weyl modules). *Let λ be a dominant weight. For any G-module M, there is a natural isomorphism*

$$\mathrm{Hom}_G(\Delta_\lambda, M) \cong \mathrm{Hom}_{B^+}(\lambda, M).$$

In particular, if M has highest weight λ, then there is a natural map from $\Delta_{\lambda_\mathbb{Z}} \otimes_\mathbb{Z} M_\lambda$ to M, its kernel has lower weights, and λ is not a weight of its cokernel.

Proof. By the tensor identity Proposition 12: $\mathrm{ind}_{B^+}^G(-\lambda \otimes M) \cong \mathrm{ind}_{B^+}^G(-\lambda) \otimes M$. Thus:

$$\mathrm{Hom}_G(\Delta_\lambda, M) = \mathrm{Hom}_G(\mathbf{k}, \mathrm{ind}_{B^+}^G(-\lambda) \otimes M) = \mathrm{Hom}_{B^+}(\mathbf{k}, -\lambda \otimes M) = \mathrm{Hom}_{B^+}(\lambda, M).$$

If M has highest weight λ, $M_\lambda = \mathrm{Hom}_{B^+}(\lambda, M)$. Following the maps, the second part follows from Proposition 15. $\square$

3.3. **Notations.** We now recall the notations from [12, §2.2]. Let the *Grosshans height function* $\mathrm{ht} : X(T) \to \mathbb{Z}$ be defined by:

$$\mathrm{ht}\gamma = \sum_{\alpha>0} \langle \gamma, \alpha^\vee \rangle.$$

For a G-module M, let $M_{\leq i}$ denote the largest G-submodule with weights λ that all satisfy: $\mathrm{ht}\lambda \leq i$. Similarly define $M_{<i} = M_{\leq i-1}$. For instance, $M_{\leq 0} = M^G$. We call the filtration $0 \subseteq M_{\leq 0} \subseteq M_{\leq 1} \cdots$ the *Grosshans filtration*, and we call its associated graded the *Grosshans graded* $\mathrm{gr}M$ of M. We put: $\mathrm{hull}_\nabla(\mathrm{gr}M) = \mathrm{ind}_B^G M^{U^+}$.

Let A be a $\mathbf{k}$-algebra on which G acts rationally through $\mathbf{k}$-algebra automorphisms. The Grosshans graded algebra $\mathrm{gr}A$ in degree i is thus:

$$\mathrm{gr}_i A = A_{\leq i}/A_{<i}.$$

3.4. **Erratum.** When $\mathbf{k}$ is a field, one knows that $\mathrm{gr}A$ embeds in a good filtration hull, which Grosshans calls R in [8], and which we call $\mathrm{hull}_\nabla(\mathrm{gr}A)$:

$$\mathrm{hull}_\nabla(\mathrm{gr}A) = \mathrm{ind}_B^G A^{U^+}.$$

When $\mathbf{k}$ is a field of positive characteristic p, it was shown by Mathieu [16, Key Lemma 3.4] that this inclusion is power-surjective: for every $b \in \mathrm{hull}_\nabla(\mathrm{gr}A)$, there is an r so that b^{p^r} lies in the subalgebra $\mathrm{gr}A$.

This result's exposition in [12, Lemma 2.3] relies on [11, Sublemma A.5.1]. Frank Grosshans has pointed out that the proof of this sublemma is not convincing beyond the reduction to the affine case. Later A. J. de Jong actually gave a counterexample to the reasoning. The result itself is correct and has been used by others. Therefore we take this opportunity to give a corrected treatment. Mathieu's result is generalized to an arbitrary base $\mathbf{k}$ in Section 4.

Proposition 17. *Let $\mathbf{k}$ be an algebraically closed field of characteristic $p > 0$. Let both A and B be of finite type over $\mathbf{k}$, with B finite over A. Put $Y = \mathrm{Spec}(A)$, $X = \mathrm{Spec}(B)$. Assume $X \to Y$ gives a bijection between $\mathbf{k}$ valued points. Then for all $b \in B$ there is an m with $b^{p^m} \in A$.*

Proof. The result follows easily from [15, Lemma 13]. We shall argue instead by induction on the Krull dimension of A.

Say B as an A-module is generated by d elements $b_1, \ldots, b_d$. Let $\mathfrak{p}_1, \ldots \mathfrak{p}_s$ be the minimal prime ideals of A.

Suppose we can show that for every i, j we have $m_{i,j}$ so that $b_j^{p^{m_{i,j}}} \in A + \mathfrak{p}_i B$. Then for every i we have m_i so that $b^{p^{m_i}} \in A + \mathfrak{p}_i B$ for every $b \in B$. Then $b^{p^{m_1+\cdots m_s}} \in A + \mathfrak{p}_1 \cdots \mathfrak{p}_s B$ for every $b \in B$. As $\mathfrak{p}_1 \cdots \mathfrak{p}_s$ is nilpotent, one finds m with $b^{p^m} \in A$ for all $b \in B$. The upshot is that it suffices to prove the sublemma for the inclusion $A/\mathfrak{p}_i \subset B/\mathfrak{p}_i B$. [It is an inclusion because there is a prime ideal $\mathfrak{q}_i$ in B with $A \cap \mathfrak{q}_i = \mathfrak{p}_i$.] Therefore we further assume that A is a domain.

Let $\mathfrak{r}$ denote the nilradical of B. If we can show that for all $b \in B$ there is m with $b^{p^m} \in A + \mathfrak{r}$, then clearly we can also find a u with $b^{p^u} \in A$. So we may as well replace $A \subset B$ with $A \subset B/\mathfrak{r}$ and assume that B is reduced. But then at least one component of $\mathrm{Spec}(B)$ must map onto $\mathrm{Spec}(A)$, so bijectivity implies there is only one component. In other words, B is also a domain.

Choose t so that the field extension $\mathrm{Frac}(A) \subset \mathrm{Frac}(AB^{p^t})$ is separable. (So it is the separable closure of $\mathrm{Frac}(A)$ in $\mathrm{Frac}(B)$.) As $X \to \mathrm{Spec}(AB^{p^t})$ is also bijective, we have that $\mathrm{Spec}(AB^{p^t}) \to \mathrm{Spec}(A)$ is bijective. It clearly suffices to prove the proposition for $A \subset AB^{p^t}$. So we replace B with AB^{p^t} and further assume that $\mathrm{Frac}(B)$ is separable over $\mathrm{Frac}(A)$.

Now $X \to Y$ has a degree which is the degree of the separable field extension. There is a dense subset U of Y so that this degree is the number of elements in the inverse image of a point of U. [Take a primitive element of the field extension, localize to make its minimum polynomial monic over A, invert the discriminant.] Thus the degree must be one because of bijectivity.

So we must now have that $\mathrm{Frac}(B) = \mathrm{Frac}(A)$. Let Let $\mathfrak{c}$ be the conductor of $A \subset B$. So $\mathfrak{c} = \{ b \in B \mid bB \subset A \}$. We know it is nonzero. If it is the unit ideal then we are done. Suppose it is not. By induction applied to $A/\mathfrak{c} \subset B/\mathfrak{c}$ (we need the induction hypothesis for the original problem without any of the intermediate simplifications) we have that for each $b \in B$ there is an m so that $b^{p^m} \in A + \mathfrak{c} = A$. $\square$

3.5. This subsection prepares the ground for the proof of the theorems in Section 4. We start with the ring of invariants $\mathbf{k}[G/U]$ of the action of U by right translation on $\mathbf{k}[G]$.

Lemma 18. *The $\mathbf{k}$-algebra $\mathbf{k}[G/U]$ is finitely generated.*

Proof. We have:

$$k[G/U] = \bigoplus_{\lambda \in X(T)} k[G/U]_{-\lambda} = \bigoplus_{\lambda \in X(T)} (k[G] \otimes \lambda)^B = \bigoplus_{\lambda \in X(T)} \nabla_\lambda.$$

By Proposition 11, this equals the sum $\oplus_\lambda \nabla_\lambda$ over dominant weights λ only. When G is simply connected, every fundamental weight is a weight, and the monoid of dominant λ is finitely generated. In general, some multiple of a fundamental weight is in $X(T)$ and there are only finitely many dominant weights modulo these multiples. So the monoid is still finitely generated by Dickson's Lemma [4, Ch. 2 Thm. 7]. The maps $\nabla_\lambda \otimes \nabla_\mu \to \nabla_{\lambda+\mu}$ are surjective for dominant λ, μ, because this is so over $\mathbb{Z}$, by base change and surjectivity for fields [10, II, Proposition 14.20]. This implies the result. $\square$

In the same manner one shows:

Lemma 19. *If the $\mathbf{k}$-algebra A^U is finitely generated, so is* $\mathrm{hullgr} A = \mathrm{ind}_B^G A^U$. $\square$

Lemma 20. *Suppose $\mathbf{k}$ is Noetherian. If the $\mathbf{k}$-algebra A is finitely generated, then so is A^U.*

Proof. By the transfer principle [7, Ch. Two]:

$$A^U = \mathrm{Hom}_U(k, A) = \mathrm{Hom}_G(k, \mathrm{ind}_U^G A) = (A \otimes k[G/U])^G.$$

Now apply Lemma 18 and Theorem 3. $\square$

Lemma 21. *If M is a G-module, there is a natural injective map*

$$\mathrm{gr} M \hookrightarrow \mathrm{hull}_\nabla(\mathrm{gr} M) = \mathrm{ind}_B^G M^{U^+}.$$

Proof. By Lemma 14, the weights of M^{U^+} are dominant. If one of them, say λ, has Grosshans height i, the universal property of Weyl modules (Proposition 16) shows that $(M^{U^+})_\lambda$ is contained in a G-submodule with weights that have not a larger Grosshans height. So the weight space $(M^{U^+})_\lambda$ is contained in $M_{\leq i}$, but not $M_{<i}$. We conclude that the T-module $\oplus_i (\mathrm{gr}_i M)^{U^+}$ may

be identified with the T-module M^{U^+}. It remains to embed $\mathrm{gr}_i M$ into $\mathrm{ind}_B^G(\mathrm{gr}_i M)^{U^+}$. The T-module projection of $\mathrm{gr}_i M$ onto $(\mathrm{gr}_i M)^{U^+}$ may be viewed as a B-module map, and then, it induces a G-module map $\mathrm{gr}_i A \to \mathrm{ind}_B^G((\mathrm{gr}_i A)^{U^+})$, which restricts to an isomorphism on $(\mathrm{gr}_i A)^{U^+}$ by Proposition 15. So its kernel has weights with lower Grosshans height, and must therefore be zero. □

In the light of Lemma 21, one may write:

Definition 22. A G-module M has *good Grosshans filtration* if the embedding of $\mathrm{gr}M$ into $\mathrm{hull}_\nabla(\mathrm{gr}M)$ is an isomorphism.

It is worth recording the following characterization, just like in the case where $\mathbf{k}$ is a field.

Proposition 23 (Cohomological criterion)**.** *For a G-module M, the following are equivalent.*

(i) *M has good Grosshans filtration.*
(ii) *$H^1(G, M \otimes \mathbf{k}[G/U])$ vanishes.*
(iii) *$H^n(G, M \otimes \mathbf{k}[G/U])$ vanishes for all positive n.*

Proof. Let M have good Grosshans filtration. We have to show that $M\otimes\mathbf{k}[G/U]$ is acyclic. First, for each integer i, $\mathrm{gr}_i M \otimes \mathbf{k}[G/U]$ is a direct sum of modules of the form $\mathrm{ind}_B^G\lambda \otimes \mathrm{ind}_B^G\mu \otimes N$, where G acts trivially on N. Such modules are acyclic by [10, B.4] and the universal coefficient theorem. As each $\mathrm{gr}_i M \otimes \mathbf{k}[G/U]$ is acyclic, so is each $M_{\le i} \otimes \mathbf{k}[G/U]$, and thus $M \otimes \mathbf{k}[G/U]$ is acyclic.

Conversely, suppose that M does not have good Grosshans filtration. Chose i so that $M_{<i}$ has good Grosshans filtration, but $M_{\le i}$ does not. Choose λ so that $\mathrm{Hom}(\Delta_\lambda, \mathrm{hull}(\mathrm{gr}_i M)/\mathrm{gr}_i M)$ is nonzero. Note that λ has Grosshans height below i. As $\mathrm{Hom}(\Delta_\lambda, \mathrm{hull}(\mathrm{gr}_i M))$ vanishes, $\mathrm{Ext}^1_G(\Delta_\lambda, \mathrm{gr}_i M) = H^1(G, \mathrm{gr}_i M \otimes \nabla_\lambda)$ does not. Since $M_{<i} \otimes \mathbf{k}[G/U] = \oplus_{\mu \text{ dominant}} M_{<i} \otimes \nabla_\mu$ is acyclic, $H^1(G, M_{\le i} \otimes \nabla_\lambda)$ is nonzero as well. Now use that $\mathrm{Hom}(\Delta_\lambda, M/M_{\le i})$ vanishes, and conclude that $H^1(G, M \otimes \mathbf{k}[G/U])$ does not vanish. □

4. Grosshans graded, Grosshans hull and powers

4.1. When G be a semisimple group over a field $\mathbf{k}$, Grosshans has introduced a filtration on G-modules. As recalled in Section 3.3, it is the filtration associated to the function defined on $X(T)$ by: $\mathrm{ht}\gamma = \sum_{\alpha>0}\langle\gamma, \alpha^\vee\rangle$. Grosshans has proved some interesting results about its associated graded, when the G-module is a $\mathbf{k}$-algebra A with rational G action. We now show how these results generalize to an arbitrary Noetherian base $\mathbf{k}$, and we draw some conclusions about $H^*(G, A)$. All this suggests that the finite generation conjecture of [12] (see also [13]) deserves to be investigated in the following generality.

Problem. *Let $\mathbf{k}$ be a Noetherian ring and let G be a Chevalley group scheme over $\mathbf{k}$. Let A be a finitely generated $\mathbf{k}$-algebra on which G acts rationally through algebra automorphisms. Is the cohomology ring $\mathrm{H}^*(G, A)$ a finitely generated $\mathbf{k}$-algebra?*

Let $\mathbf{k}$ be an arbitrary commutative ring.

Theorem 24 (Grosshans hull and powers)**.** *The natural embedding $\mathrm{gr}M \subseteq \mathrm{hull}_\nabla(\mathrm{gr}M)$ is power surjective.*

This will then be used to prove:

Theorem 25 (Grosshans hull and finite generation)**.** *If the ring $\mathbf{k}$ is Noetherian, then the following are equivalent.*

(i) *The $\mathbf{k}$-algebra A is finitely generated;*

(ii) *For every standard parabolic P, the* $\mathbf{k}$*-algebra of invariants* $A^{R_u(P)}$ *is finitely generated;*
(iii) *The* $\mathbf{k}$*-algebra* $\mathrm{gr}A$ *is finitely generated;*
(iv) *The* $\mathbf{k}$*-algebra* $\mathrm{hull}_\nabla(\mathrm{gr}A)$ *is finitely generated.*

Remark 26. Consider a reductive Chevalley group scheme G. As the Grosshans height is formulated with the help of coroots $\alpha^\vee$, only the semisimple part of G is relevant for it. But of course everything is compatible with the action of the center of G also. We leave it to the reader to reformulate our results for reductive G. We return to the assumption that G is semisimple.

Theorem 27. *Let A be a finitely generated* $\mathbf{k}$*-algebra. If* $\mathbf{k}$ *is Noetherian, there is a positive integer n so that:*

$$n\mathrm{hull}_\nabla(\mathrm{gr}A) \subseteq \mathrm{gr}A.$$

In particular $H^i(G, \mathrm{gr}A)$ is annihilated by n for positive i.

This is stronger than the next result.

Theorem 28 (generic good Grosshans filtration)**.** *Let A be a finitely generated* $\mathbf{k}$*-algebra. If* $\mathbf{k}$ *is Noetherian, there is a positive integer n so that $A[1/n]$ has good Grosshans filtration. In particular $H^i(G, A) \otimes \mathbb{Z}[1/n] = H^i(G, A[1/n])$ vanishes for positive i.*

Remark 29. Of course $A[1/n]$ may vanish altogether, as we are allowed to take the characteristic for n, when that is positive.

4.2. We start with a crucial special case. Let $\mathbf{k} = \mathbb{Z}$. Let $\lambda \in X(T)$ be dominant. Let S' be the graded algebra with degree n part:

$$S'_n = \nabla_{n\lambda} = \Gamma(G/B, \mathcal{L}(n\lambda)).$$

Let us view Δ_λ as a submodule of ∇_λ with common λ weight space (the 'minimal admissible lattice' embedded in the 'maximal admissible lattice'). Let S be the graded subalgebra generated by Δ_λ in the graded algebra S'. If we wish to emphasize the dependence on λ, we write $S'(\lambda)$ for S', $S(\lambda)$ for S. Consider the map

$$G/B \to \mathbb{P}_{\mathbb{Z}}(\Gamma(G/B, \mathcal{L}(\lambda))^\#)$$

given by the 'linear system' ∇_λ on G/B. The projective scheme $\mathrm{Proj}(S')$ corresponds with its image, which, by direct inspection, is isomorphic to G/P, where P is the stabilizer of the weight space with weight $-\lambda$ of $\nabla_\lambda^\#$. The inclusion $\phi : S \hookrightarrow S'$ induces a morphism from an open subset of $\mathrm{Proj}(S')$ to $\mathrm{Proj}(S)$.This open subset is called $G(\phi)$ in [EGA II, 2.8.1].

Lemma 30. *The morphism* $\mathrm{Proj}(S') \to \mathrm{Proj}(S)$ *is defined on all of* $G/P = \mathrm{Proj}(S')$.

Proof. As explained in [EGA II, 2.8.1], the domain $G(\phi)$ contains the principal open subset $D_+(s)$ of $\mathrm{Proj}(S')$ for any $s \in S_1$. Consider in particular a generator s of the λ weight space of ∇_λ. It is an element in S_1, and, by Lemma 15, it generates the free $\mathbf{k}$-module $\Gamma(P/P, \mathcal{L}(\lambda))$. Thus, the minimal Schubert variety P/P is contained in $D_+(s)$. We then conclude by homogeneity: s is also U^+ invariant, so in fact the big cell $\Omega = U^+P/P$ is contained in $D_+(s)$, and the domain $G(\phi)$ contains the big cell Ω. Then it also contains the Weyl group translates $w\Omega$, and thus it contains all of G/P. □

Lemma 31. *S' is integral over S.*

Proof. We also put a grading on the polynomial ring $S'[z]$, by assigning degree one to the variable z. One calls $\mathrm{Proj}(S'[z])$ the projective cone of $\mathrm{Proj}(S')$ [EGA II, 8.3]. By [EGA II, 8.5.4], we get from Lemma 30 that $\hat{\Phi} : \mathrm{Proj}(S'[z]) \to \mathrm{Proj}(S[z])$ is everywhere defined. Now by [EGA II, Th (5.5.3)], and its proof, the maps $\mathrm{Proj}(S'[z]) \to \mathrm{Spec}\mathbb{Z}$ and $\mathrm{Proj}(S[z]) \to \mathrm{Spec}\mathbb{Z}$ are proper and separable, so $\hat{\Phi}$ is proper by [EGA II, Cor (5.4.3)]. But now the principal open subset $D_+(z)$ associated to z in $\mathrm{Proj}(S'[z])$ is just $\mathrm{Spec}(S')$, and its inverse image is the principal open subset associated to z in $\mathrm{Proj}(S[z])$, which is $\mathrm{Spec}(S)$ (compare [EGA II, 8.5.5]). So $\mathrm{Spec}(S) \to \mathrm{Spec}(S')$ is proper, and S is a finitely generated S'-module by [EGA III, Prop (4.4.2)]. $\square$

Lemma 32. *There is a positive integer t so that tS' is contained in S.*

Proof. Clearly $S' \otimes \mathbb{Q} = S \otimes \mathbb{Q}$, so the result follows from Lemma 31. $\square$

If p is a prime number, then it is a result of Mathieu [16, Key Lemma 3.4] that, for every element b of S'/pS', there is a positive r so that $b^{p^r} \in (S + pS')/(pS') \subseteq S'/pS'$.

Lemma 33. *For each b in S', there is a positive integer s so that b^s is in S.*

Proof. We may compute modulo tS', with t from Lemma 32. For every prime p dividing t of Lemma 32 we may [16, Key Lemma 3.4], replace b with a power that lies inside $S + pS'$. So if $p_1,\ldots,p_m$ are the primes dividing t, we can arrange that b lies in the intersection of the $S + p_iS'$, which is $S + p_1 \cdots p_mS'$. Now by taking repeated $p_1 \cdots p_m$-th powers, one pushes it in $S + (p_1 \cdots p_m)^nS'$ for any positive n, eventually in $S + tS' \subseteq S$. $\square$

4.3. Let us now return to a general ring $\mathbf{k}$ and $\mathbf{k}$-algebra A, and let us consider the inclusion $\mathrm{gr}A \hookrightarrow \mathrm{hull}_\nabla(\mathrm{gr}A)$, as in Theorem 24.

Notations 34. Let λ be a dominant weight and let $b \in A^{U^+}$ be a weight vector of weight λ. Then we define $\psi_b : S'(\lambda) \otimes \mathbf{k} \to \mathrm{hull}_\nabla(\mathrm{gr}A)$ as the algebra map induced by the B-algebra map $S'(\lambda) \otimes \mathbf{k} \to A^{U^+}$ which sends the generator (choose one) of the λ weight space of ∇_λ to b.

Lemma 35. *For each c in the image of ψ_b, there is a positive integer s so that $c^s \in \mathrm{gr}A$.*

Proof. The composite of $S \otimes \mathbf{k} \to S' \otimes \mathbf{k}$ with ψ_b factors through $\mathrm{gr}A$, so this follows as in the proof of Lemma 33. $\square$

Proof of Theorem 24. For every $b \in \mathrm{hull}_\nabla(\mathrm{gr}A)$, there are $b_1,\ldots, b_s$ so that b lies in the image of $\psi_{b_1} \otimes \cdots \otimes \psi_{b_s}$. Lemmas 32, 33, 35 easily extend to tensor products. $\square$

Lemma 36. *Suppose* $\mathbf{k}$ *is Noetherian. If* $\mathrm{hull}_\nabla(\mathrm{gr}A)$ *is a finitely generated* $\mathbf{k}$*-algebra, so is* $\mathrm{gr}A$.

Proof. Indeed, $\mathrm{hull}_\nabla(\mathrm{gr}A)$ is integral over $\mathrm{gr}A$ by Theorem 24. Then it is integral over a finitely generated subalgebra of $\mathrm{gr}A$, and it is a Noetherian module over that subalgebra. $\square$

Lemma 37. *If* $\mathrm{gr}A$ *is finitely generated as a* $\mathbf{k}$*-algebra, then so is A.*

Proof. Say $j_1, \ldots, j_n$ are nonnegative integers and $a_i \in A_{\leq j_i}$ are such that the classes $a_i + A_{<j_i} \in \mathrm{gr}_{j_i}A$ generate $\mathrm{gr}A$. Then the a_i generate A. $\square$

Lemma 38. *Suppose* $\mathbf{k}$ *is Noetherian. If A^U is a finitely generated* $\mathbf{k}$*-algebra, so is A.*

Proof. Combine Lemmas 19, 36, 37. $\square$

Lemma 39. *Let P be a standard parabolic subgroup. Suppose* $\mathbf{k}$ *is Noetherian. Then A is a finitely generated* $\mathbf{k}$*-algebra if and only if $A^{R_u(P)}$ is one also.*

Proof. Let V be the intersection of U with the semisimple part of the standard Levi subgroup of P. Then $U = VR_u(P)$ and $A^U = (A^{R_u(P)})^V$. Suppose that A is a finitely generated $\mathbf{k}$-algebra. Then $A^U = (A^{R_u(P)})^V$ is one also by Lemma 20, and so is $A^{R_u(P)}$ by Lemma 38 (applied with a different group and a different algebra).

Conversely, if $A^{R_u(P)}$ is a finitely generated $\mathbf{k}$-algebra, Lemma 20 (with that same group and algebra) implies that $A^U = (A^{R_u(P)})^V$ is finitely generated, and thus A is as well, by Lemma 38. $\square$

Proof of Theorem 25. Combine Lemmas 39, 20, 19, 36, 37. □

Proof of Theorem 27. Let $\mathbf{k}$ be Noetherian and let A be a finitely generated $\mathbf{k}$-algebra. By Theorem 25, the $\mathbf{k}$-algebra $\mathrm{hull}_\nabla(\mathrm{gr}A)$ is finitely generated. So we may choose $b_1,\dots,b_s$, so that $\psi_{b_1}\otimes\cdots\otimes\psi_{b_s}$ has image $\mathrm{hull}_\nabla(\mathrm{gr}A)$. By extending Lemma 32 to tensor products, we can argue as in the proof of Lemma 35 and Theorem 24, and see that there is a positive integer n so that $n\mathrm{hull}_\nabla(\mathrm{gr}A)\subseteq\mathrm{gr}A$. Now, $\mathrm{hull}_\nabla(\mathrm{gr}A)\otimes\mathbf{k}[G/U]$ is acyclic by Proposition 23, and thus its summand $\mathrm{hull}_\nabla(\mathrm{gr}A)$ is acyclic as well. It follows that $H^i(G,\mathrm{gr}A)$ is a quotient of $H^{i-1}(G,\mathrm{hull}_\nabla(\mathrm{gr}A)/\mathrm{gr}A)$, which is annihilated by n. □

Proof of Theorem 28. Take n as in Theorem 27, and use that localization is exact. □

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